NPS-62-89-019

NAVAL POSTGRADUATE SCHOOL

Monterey, California

THEORY
OF
MULTI-FREQUENCY MODULATION (MFM)
DIGITAL COMMUNICATIONS

Paul H. Moose

Interim Report for the Period
October 1988 to March 1989

Approved for public release; distribution unlimited.

Prepared for:

Naval Postgraduate School
Monterey, CA 93943

NAVAL POSTGRADUATE SCHOOL
Monterey, CA

Rear Admiral R.C. Austin
Superintendent

Dr. Harrison Shull
Provost

Reproduction of all or part of this report is authorized.

This report was prepared in conjunction with research conducted for the Naval Oceans System Command and funded by the Naval Postgraduate School.

This report was prepared by:

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a REPORT SECURITY CLASSIFICATION UNCLASSIFIED	1b RESTRICTIVE MARKINGS
2a SECURITY CLASSIFICATION AUTHORITY 2b DECLASSIFICATION / DOWNGRADING SCHEDULE	3 DISTRIBUTION / AVAILABILITY OF REPORT Approved for public release; distribution is unlimited.
4 PERFORMING ORGANIZATION REPORT NUMBER(S) NPS 62-89-019	5 MONITORING ORGANIZATION REPORT NUMBER(S)
6a NAME OF PERFORMING ORGANIZATION Naval Postgraduate School 6b OFFICE SYMBOL (If applicable) 62Me	7a NAME OF MONITORING ORGANIZATION Naval Ocean Systems Center, Code 624
6c. ADDRESS (City, State, and ZIP Code) Monterey, CA 93943-5004	7b ADDRESS (City, State, and ZIP Code) San Diego, CA 92152
8a NAME OF FUNDING / SPONSORING ORGANIZATION Naval Postgraduate School 8b OFFICE SYMBOL (If applicable)	9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER O & MN Direct Funding
8c ADDRESS (City, State, and ZIP Code) Monterey, CA 93943-5000	10 SOURCE OF FUNDING NUMBERS PROGRAM ELEMENT NO · PROJECT NO · TASK NO · WORK UNIT ACCESSION NO

11 TITLE (Include Security Classification)

Theory of Multifrequency Modulation (MFM) Digital Communications

12 PERSONAL AUTHOR(S)

Paul H. Moose

13a TYPE OF REPORT	13b TIME COVERED	14 DATE OF REPORT (Year, Month, Day)	15 PAGE COUNT
Research	FROM 10/88 TO 3/89	1989, May 5	38

16 SUPPLEMENTARY NOTATION

17 COSATI CODES			18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	

19 ABSTRACT (Continue on reverse if necessary and identify by block number)

Multi-frequency modulation (MFM) is a new digital signal processing (DSP) oriented communications signal developed at NPS specifically for computer-to-computer communications links and information exchange networks. MFM utilizes the hardware and software of the host computers to generate and to demodulate coherent communications discrete time signals.

In this report, the theory behind MFM generation and reception is presented. Autocorrelation functions and power spectral densities of MFM signals are derived and examples presented for lowpass and bandpass white MFM sequences. The bit error rates are computed for three types of MFM: MFBPSK, MFQPSK and MF16-QAM. These modulation formats provide one, two and four bits per Hz of channel bandwidth respectively. Optimization arguments show that best system performance is obtained by using the maximum possible number of tones with the limit on the number of tones being set either by the packet length or the coherence time of the channel, whichever is shorter.

20 DISTRIBUTION / AVAILABILITY OF ABSTRACT ☐ UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT ☐ DTIC USERS	21 ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED
22a NAME OF RESPONSIBLE INDIVIDUAL Paul H. Moose	22b TELEPHONE (Include Area Code) (408) 646-2838 22c OFFICE SYMBOL 62Me

DD FORM 1473, 84 MAR 83 APR edition may be used until exhausted. All other editions are obsolete

SECURITY CLASSIFICATION OF THIS PAGE

☆ U.S. Government Printing Office 1986—606-24

THEORY
OF
MULTI-FREQUENCY MODULATION (MFM)
DIGITAL COMMUNICATIONS

Paul H. Moose

TABLE OF CONTENTS

I. INTRODUCTION	1
II. THEORY OF MULTI-FREQUENCY MODULATION	5
III. PERFORMANCE OF MFM	21
IV. DISCUSSION AND CONCLUSIONS	34
REFERENCES	36
APPENDIX 1	37
DISTRIBUTION LIST	38

LIST OF FIGURE TITLES

Figure 1 MFM Signal Packet

Figure 2 Data Structure for IDFT

Figure 3 Block Diagram of Transmitter

Figure 4 Block Diagram of Receiver

Figure 5 ACF and PSD of White Bandpass Sequence

Figure 6 ACF and PSD of Baud Repeated Four Times

Figure 7 MFM Phase Modulation Formats (Hex)

Figure 8 MFM Bit Error Probabilities vs. SNR_{NB}

I. INTRODUCTION

Computer-to-computer communications characterizes the fundamental link for message and data traffic in many modern military communications scenarios. In some cases the link is one of many in an extensive network, in others it may represent a single point-to-point communications link. Regardless, the objective is to transmit bits with a satisfactorily low rate of errors from one terminal to another.

The medium for transmitting the data may be wire or optical fiber, that is a low pass channel, or it may be radio frequency or acoustic, that is a bandpass channel. In either case, the actual signal used to carry the data bits must be a properly modulated analog signal with sufficient energy and positioned in the frequency spectrum to propagate effectively through the available medium. In RF links the signal will be further restricted to an assigned frequency slot or channel in the spectrum. For network links transmission may be restricted to assigned time slots.

What is needed is a signal modulation format that is readily adaptable to a variety of link environments, does not require special purpose MODEMS to translate between the digital and analog domains, can emulate most existing signal modulation formats as well as generate entirely new formats

and whose descriptive language is the language of Digital Signal Processing (DSP). Multi-Frequency Modulation (MFM) is such a technique and research into its implementation as a practical communications method is the subject of this report.

Classical modulation methods for bandpass channels use amplitude and/or phase to carry signal information on a carrier wave in the channel band. When the information source is a finite alphabet, as with data or quantized analog sources such as digitized speech or video, then only a finite number of signal states are needed to represent, or code the source.(The presence of one or the other of two frequencies, FSK, is also used to transmit binary data, but this is less efficient than phase shift keying, PSK,[ref 1].)

In order to accommodate multiple sources, individual channels are multiplexed together either in the frequency domain,(FDM) or time domain,(TDM) or both. With multiple users on a network, Time Domain Multiple Access, (TDMA), or Frequency Domain Multiple Access, (FDMA) is used to allow each data source the opportunity to transmit on the net.

Modulation and multiplexing equipment for communications links carrying many channels is primarily analog circuitry and is very expensive to purchase and maintain. Both commercial and military communications system developers

desire to make their equipment as digitally oriented as possible in order to increase its flexibility and to automate its testing [ref 2]. This trend is being followed at SATCOM earth stations, at military switching centers, NAVCOMSTA's for example, on ships and aboard aircraft.

It is our view that ultimate digitization is achieved by having the host computers absorb the functions of modulation and multiplexing, demultiplexing and demodulation. MFM is a modulation technique that is ideally suited to this task because its basic structure is one of time and frequency slots. In MFM the signals are directly encoded and modulated, and decoded and demodulated using digital signal

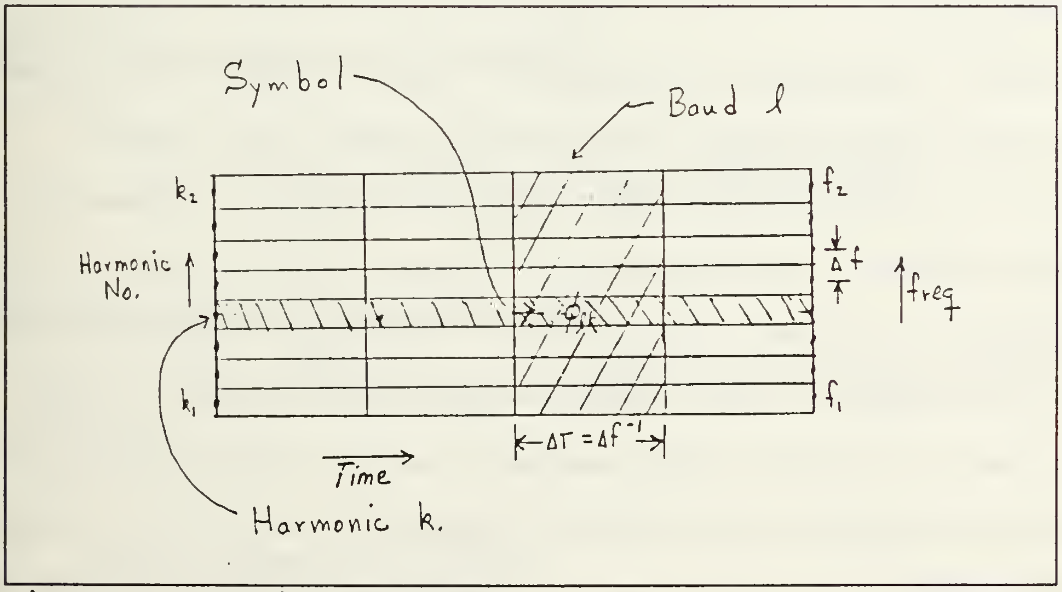

Figure 1 MFM Signal Packet

processing techniques within the host computers. Connections to the analog link are made thru D/A and A/D chips on plug-in boards over DMA I/O channels. Over eighty percent of current digital communications signals, including spread-spectrum signals, time and frequency slotted signals, plus many entirely new signal formats, are simply generated and demodulated using MFM.

In MFM, signal "packets" arbitrarily located in the frequency spectrum and in time are created as shown in Fig.1. Each packet consists of L bauds of K tones. These LK subsignals form an orthogonal signal set. Each subsignal may be independently modulated with phase and amplitude information.

From a theoretical standpoint, recent research has shown that multitone signals are more efficient than single carrier signals of the same total bandwidth in a general linear channel, particuarily if there is a deep null in the frequency response [ref 3]. From a practical point of view, we have suggested that MFQPSK signals are ideally suited for use in high data rate acoustic burst communications from moving platforms using medium frequency, relatively wideband (25% bandwidth) links [ref 4]. The appealing thing about MFM signals is the simple way they can be generated, demodulated, and equalized using standard digital signal processing algorithims which are now available for nearly all industry standard digital computers in a variety of high

level languages. Real time processing at high continuous data rates is still difficult, however improvements in this area are inevitable. A variety of array processor cards are available now as add-on cards and new designs (the NEXT computer for example) are incorporating DSP processors into the computer's basic architecture. Thus, although current technology limits direct bandpass generation of MFM signals to the tens to perhaps several hundereds of kilohertz range, generation of wideband HF radio signals (signals in the tens of MHz range) should be quite feasible within the decade. Of course, the audio range signals we can generate now are useful in acoustic communications, an application that is being pursued by us in conjuction with the Naval Ocean Systems Center in San Diego.

II. THEORY OF MULTI-FREQUENCY MODULATION

Generation and Demodulation of MFM

The following definitions are used in MFM (Please refer to Figure 1):

T: Packet length in seconds

ΔT: Baud length in seconds

k_x: Baud length in number of samples

L: Number of bauds per packet

Δt: Time between samples in seconds

$f_x=1/\Delta t$: Sampling or clock frequency for D/A and A/D conversion in Hz.

$\Delta f=1/\Delta T$: Frequency spacing (minimum) between MFM tones.

K: Number of MFM tones.

Since $\Delta t=\Delta T/k_x$, the sample frequency $f_x=k_x\Delta f$. Consequently, there are a maximum of $k_x/2$ tones spaced at intervals of Δf Hz between dc and less than $f_x/2$, the Nyquist frequency, that can carry amplitude and phase information during each baud. Some, or many, of the tones may not be used (or equivalently have amplitudes of zero) during any or all bauds of the packet. For example, to generate bandpass signals between frequencies f_1 and f_2, only tones between harmonics $k_1=f_1/\Delta f$ and $k_2=f_2/\Delta f$ will be allowed non-zero amplitudes. Here, the maximum number of tones will be $K=k_2-k_1+1$ and the signal bandwidth will be $W=K*\Delta f$. Note that the time bandwidth product of the entire signal packet, $TW=L\Delta T*K\Delta f=LK$, is equal to LK, the total number of symbols that can be sent in the packet.

Let us now provide a mathematical description of the signal packet. Let the l^{th} baud of the transmit signal be described by;

$$x_l(u) = \sum_k x_{lk}(u) \qquad (1)$$

where,

$$x_{lk}(u) = A_{lk}\cos(2\pi k\Delta f u + \phi_{lk}) \ ; \ 0\leq u\leq \Delta T. \qquad (2)$$

Here, u is time referenced to the beginning of the baud. Actual real time is $t=t_0+l\Delta T+u$ where t_0 is the time of initiation of the 0^{th} baud, i.e., the beginning time of the packet. Now the discrete time signal corresponding to the l^{th} baud is found by sampling (1) and (2) at the sampling intervals $\Delta t=1/f_x$ and is given by;

$$x_l(n) = \sum_k x_{lk}(n) \qquad (3)$$

where,

$$x_{lk}(n) = A_{lk}\cos(2\pi kn/k_x + \phi_{lk}) \; ; \; 0\leq n\leq k_x-1. \qquad (4)$$

Here, n is discrete time referenced to the beginning of the baud. Note that, in general, k may take on all integer values between 0 and $k_x/2$. We will refer to k as the "harmonic number" of the MFM tone of frequency $k\Delta f$. It is useful to note that a baud interval, i.e., a time ΔT, contains <u>exactly</u> k cycles of tone k. Thus, adjacent tones differ by one in the number of cycles they make during a baud.

Consider the k_x point Discrete Fourier Transform (DFT) of (3). It is given by;

$$X_l(k') = \sum_k X_{lk}(k') \qquad (5)$$

where,

$$X_{1k}(k') = \tfrac{1}{2}k_x A_{1k}\{\exp(-j\phi_{1k})\delta(k'-k)$$

$$+ \exp(j\phi_{1k})\delta(k'-(k_x-k))\} \;;\; 0 \le k' \le k_x-1. \qquad (6)$$

Thus, it is apparent that the discrete time signal (3) is given by the k_x point Inverse Discrete Fourier Transform (IDFT) of (5), namely;

$$x_1(n) = \mathrm{IDFT}[X_1(k)] \qquad (7)$$

where from (6) it is clear that,

$$\begin{aligned} X_1(k) &= \tfrac{1}{2}k_x A_{1k}\exp(-j\phi_{1k}) \;;\quad 0 \le k < k_x/2 \\ X_1(k_x-k) &= \tfrac{1}{2}k_x A_{1k}\exp(j\phi_{1k}) \;;\quad 0 < k < k_x/2 \end{aligned} \qquad (8)$$

To summarize, the 1^{th} baud is generated by taking the IDFT of a complex valued array of length k_x. The first half of the array is loaded with the amplitudes and phases of the tones to be included in the MFM signal at the corresponding harmonic numbers. The second half of the array is loaded with the complex conjugate of the values in the first half of the array at the image harmonics (the image harmonic of k is k_x-k). This data structure is shown for k_x=16 in Fig. 2. The complex conjugate symmetry about the midpoint insures that the IDFT, also an array of k_x points, will be real

valued. The values in this array, $x_l(n)$, are the discrete time samples of the analog transmit signal. Clocking these samples out thru a D/A converter (they are stored as binary numbers in the computer) at f_x samples per second completes the generation of the l^{th} baud. The total signal packet is generated by an L-fold repetition of this process.

k	Re{X(k)}	Im{X(k)}
0	0	0
1	0	0
2	0	0
3	XR3	XI3
4	XR4	XI4
5	XR5	XI5
6	0	0
7	0	0
8	0	0
9	0	0
10	0	0
11	XR5	-XI5
12	XR4	-XI4
13	XR3	-XI3
14	0	0
15	0	0

Figure 2 Data Structure for IDFT

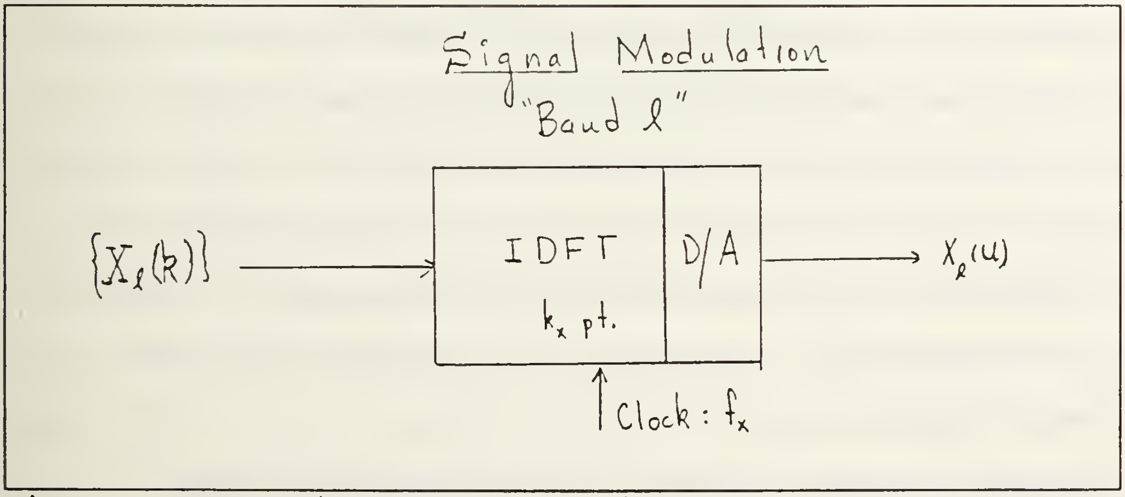

Figure 3 Block Diagram of Transmitter.

A block diagram of the transmitter is shown in Fig. 3. At NPS we have implemented this using an IBM PC/AT as a host computer. Samples may be clocked out at any clock rate up to 250 KHz; this limit is set by the maximum byte transfer rate of the AT's DMA channel [ref 5]. The maximum packet size is 61440 samples; this limit is set, currently, at one sector of the AT's memory. We have been using this system to generate bandpass signals in the 10-14 KHz band and 16-20 KHz band using a clock rate of 61440 samples per second. The relevant signal parameters are given in Appendix 1.

Demodulation of MFM is accomplished by a process inverse to its generation. Given the time domain signal at the receiver $x_l(u)$ on the interval $0 \le u \le \Delta T$: The signal is sampled at f_x samples per second and converted to digital format with an A/D converter. The k_x real values thus obtained are loaded into a k_x point complex valued array (the imaginary parts are set to zero). This becomes the array $x_l(n)$ for the l^{th} baud. Its k_x point DFT yields the complex valued array $X_l(k)$ containing, in its first half, the amplitude and phase modulation information, A_{lk} and ϕ_{lk}, of the K harmonics employed in the generation of the transmit signal.

Notice that the demodulation operation is linear (ignoring for the moment the non-linear nature of the A/D conversion). Also notice that only K out of k_x complex values are retained for decoding. The upper half of the

array is redundant; it contains the complex conjugates of the values at the image frequencies. Also, non-used harmonics in the lower half of the array will, in the absence of noise, contain zero values and can be discarded. When they do contain noise values, their discard is equivalent to filtering out the unused portion of the spectrum at the ouput of the demodulation stage.

Later in this report, it will be shown that DFT demodulation of the signal is equivalent to correlation, or matched filter, reception for each of the K tones. It will also be shown that the tones are all orthogonal; in discrete time on the interval $0 \le n \le k_x - 1$, and in continuous time on the interval $0 \le u \le \Delta T$. Of course, signals in different bauds are orthogonal, since they do not overlap in (real) time. Thus, the K tones of the L bauds form a set of LK orthogonal signals so the response of a matched filter, or correlator, to any of the tones (regardless of its modulation) other than the one to which it is matched is zero. The fact that the receiver is linear, and a matched filter assures that it is optimal for demodulation of MFM, that is it maximizes signal-to-noise ratio, in the presence of additive white noise. This is so in spite of unknown link attenuation at each of the frequencies. We note that the channel need only have constant attenuation over narrow bands corresponding to the bandwidths of the individual tones, Δf, and not over the entire bandwidth W. Thus the demodulation is still optimal

for a wideband signal propagating thru a channel with substantial variation in gain across the band. A block diagram of the MFM receiver is shown in Fig.4.

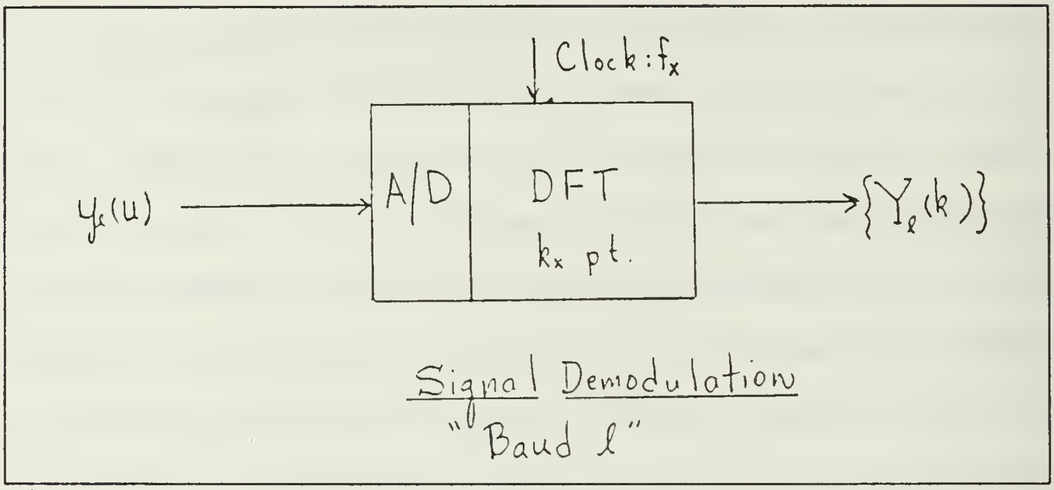

Figure 4 Block diagram of receiver

Properties of MFM Signals

In this section, a number of important properties of MFM signals will be presented. In almost every case we shall be referring to a single baud, say the 1^{th} baud, so, unless required for clarity of presentation, the subscript 1 will be dropped.

a. Orthogonality of subsignals.

In continuous time;

$$\int_0^{\Delta T} x_k(u)\,x_i(u)\,du = \begin{cases} 0 & ;\ k \neq i \\ \frac{1}{2}(A_k)^2\,\Delta T & ;\ k = i \end{cases} \qquad (9)$$

and in discrete time;

$$\sum_{n} x_k(n)x_i(n) = \begin{cases} 0 & ; \; k \neq i \\ \frac{1}{2}(A_k)^2 k_x & ; \; k = i. \end{cases} \qquad (10)$$

Eqns (9) and (10) are easily established using (2) and (4).

b. Autocorrelation function (acf) and power spectral density (psd) of MFM signals.

The circular (or periodic) acf of an MFM signal is defined as;

$$r_x(p) = \sum_{n} x(n)x(n \oplus p) \quad ; \; 0 \leq p \leq k_x - 1 \qquad (11)$$

where $\oplus$ signifies a left circular shift. Notice that a circular shift of any of the harmonic components of x(n) is still a harmonic component of the same frequency but with a different phase shift. Combine this observation with the orthogonality property (10) and (11) is easily evaluated to be ;

$$r_x(p) = \sum_{k} r_k(p) \qquad (12)$$

with,

$$r_k(p) = \tfrac{1}{2}(A_k)^2 k_x \cos(2\pi kp/k_x). \qquad (13)$$

The acf is, as expected, independent of any phase modulation on the tones; however, it is modified by amplitude moduation.

It is well known that [ref 6] ;

$$DFT[r_x(p)] = X^*(k)X(k) \qquad (14)$$

and the discrete psd is defined as,

$$S_x(k) = X^*(k)X(k)/k_x \qquad (15)$$

From (8) it is seen that,

$$S_x(k) = S_x(k_x-k) = \tfrac{1}{4}(A_k)^2 k_x \;; \quad 0 \le k < k_x/2. \qquad (16)$$

Two interesting, and important examples, are provided by white lowpass and white bandpass sequences. The white lowpass sequence is defined by;

$$A_k = \begin{cases} A & ; \ 0 \le k \le K-1 \\ 0 & ; \ K \le k \le k_x/2 \end{cases} \qquad (17)$$

Its acf is determined by evaluting (13) to be;

$$r_x(p) = \tfrac{1}{2}A^2 k_x \cos\{\pi(K-1)p/k_x\} \frac{\sin(\pi Kp/k_x)}{\sin(\pi p/k_x)} \qquad (18)$$

The white bandpass sequence is defined by;

$$A_k = \begin{cases} A & ; \ k_1 \le k \le k_2 \\ \\ 0 & ; \ \text{other } k \text{ between } 0 \text{ and } k_x/2. \end{cases} \tag{19}$$

Recalling that the number of harmonics $K=k_2-k_1+1$ and defining a midband harmonic number $k_0=(k_2+k_1)/2$ then (13) can also be evaluated in closed form. For an even number of harmonics in the sum (K an even number), the acf of the white bandpass sequence is given by;

$$r_x(p) = \tfrac{1}{2}A^2 k_x \cos\{2\pi k_0 p/k_x\} \frac{\sin(\pi K p/k_x)}{\sin(\pi p/k_x)} \tag{20}$$

Figure 5 shows the acf and psd of a white bandpass sequence with a k_x of 128, a k_1 of 47 and a k_2 of 54 (K=8, k_0=50.5). Note the nulls in the envelope of the acf occur at multiples of $p=k_x/K$. In Fig. 5 this is p=16 (as it is for all the signals in Appendix 1).

To summarize, the circular acf of MFM signals is given by (13) for general amplitude modulated signals, but simplifies to (18) for white lowpass sequences and to (20) for white bandpass sequences.

For test purposes, or synchronization, we may elect to repeat a signal for several bauds. Suppose x(n) is repeated

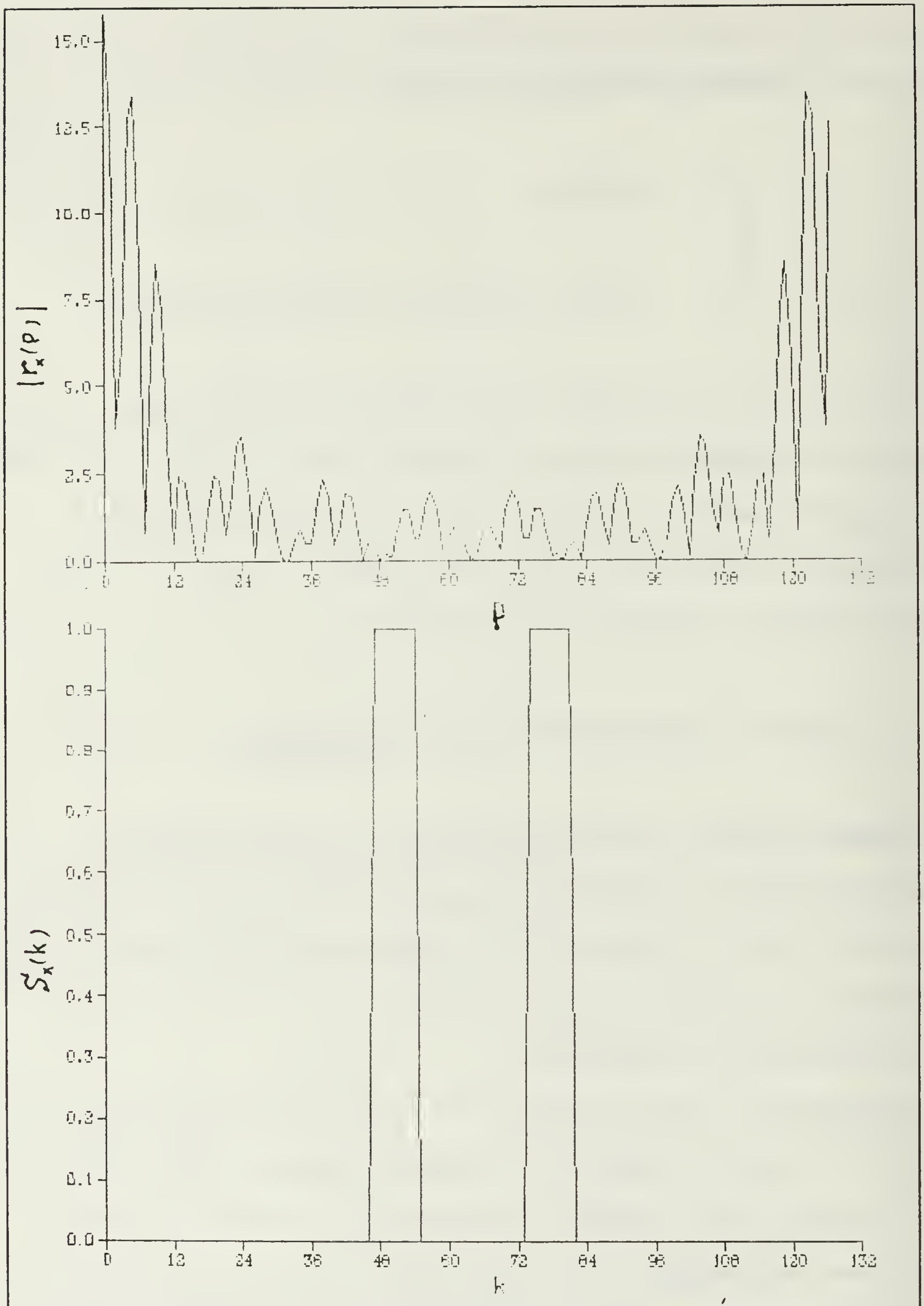

Figure 5 ACF and PSD of a white bandpass sequence.

L' times; it will have $L'k_x$ samples points. Its circular acf is given by;

$$r_x(p) = \sum_n x(n)x(n \oplus p) \quad ; \quad 0 \le p \le L'k_x - 1 \tag{21}$$

but the harmonics are still orthogonal over the repeated interval so that;

$$r_x(p) = \sum_k r_k(p) \quad ; \quad 0 \le p \le L'k_x - 1 \tag{22}$$

with,

$$r_k(p) = \tfrac{1}{2}(A_k)^2 L'k_x \cos(2\pi kp/k_x). \tag{23}$$

Note that the acf has L' times the amplitude of the acf of a single baud, and since it is still periodic with period k_x, there are peaks every k_x points. For signals with K constant amplitude harmonics, such as the white sequences described above, there will be periodic peaks of amplitude;

$$r_x(mk_x) = A^2 L'k_x K/2 \quad ; \quad 0 \le m \le L'-1 \tag{24}$$

An acf and psd for the signal of Fig. 5 repeated four times is shown in Fig. 6.

c. Matched filter (correlator) detection of MFM.

MFM signals may be further resolved into their quadrature

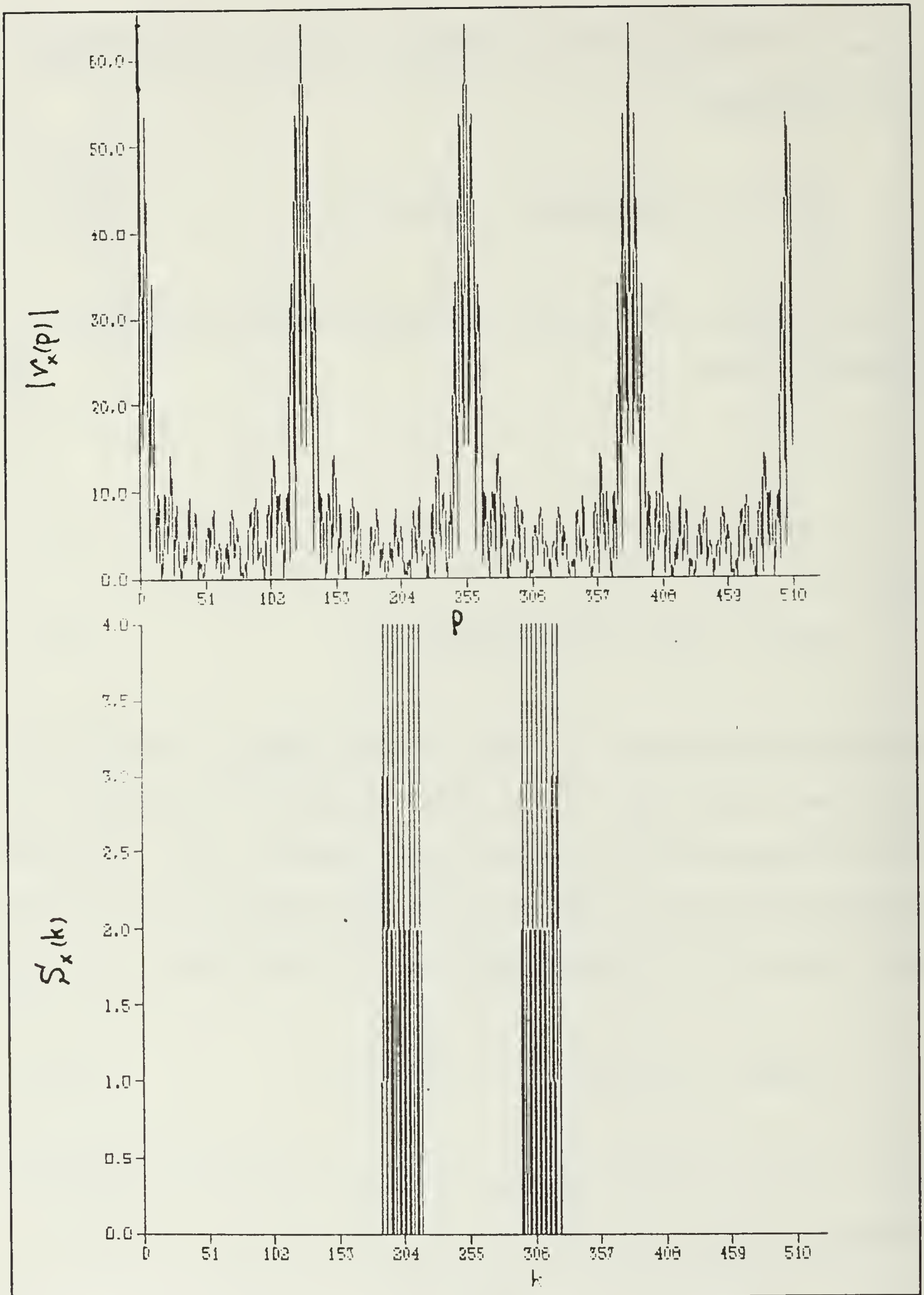

Figure 6 Acf and psd of baud repeated four times.

components. That is;

$$x_k(n) = x_{Ik}(n) + x_{Qk}(n) \tag{25}$$

where,

$$x_{Ik}(n) = \mathrm{Re}[X(k)]\cos(2\pi kn/k_x)/k_x$$

and, (26)

$$x_{Qk}(n) = -\mathrm{Im}[X(k)]\sin(2\pi kn/k_x)/k_x$$

with $X(k) = \frac{1}{2}k_x A_k \exp(j\phi_k)$ the complex DFT coefficient of the k^{th} harmonic. Note that $x_{Ik}(n)$ and $x_{Qk}(n)$ are orthogonal over a baud interval (or intervals) and they are orthogonal to the in-phase and quadrature components of all other harmonics too. So, when resolved into quadrature components, the MFM signal space actually contains 2K orthogonal signals in each baud or 2KL orthogonal signals in a packet.

The impulse response for a filter matched to the k^{th} tone of the l^{th} baud (we are still supressing l in our notation) consists in fact of a pair of filters; one filter is matched to the in-phase signal and one to the quadrature signal. Their impulse responses are given by;

$$h_{Ik}(n) = C_I x_{Ik}(k_x - n) \quad ; \quad 0 \le n \le k_x - 1 \tag{27}$$

for the in-phase component and,

$$h_{Qk}(n) = C_Q x_{Qk}(k_x - n) \quad ; \quad 0 \le n \le k_x - 1 \tag{28}$$

for the quadrature component. C_I and C_Q are arbitrary constants. The output of the discrete time matched filter pair, when excited with x(n), the 1^{th} signal baud, is given by the convolutions;

$$y_{Ik}(n) = h_{Ik}(n) * x(n)$$

and, (29)

$$y_{Qk}(n) = h_{Qk}(n) * x(n).$$

Evaluating these at the end of the baud, i.e., at $n=k_x$, and choosing C_I and C_Q to make the filters independent of the modulation yields;

$$y_{Ik}(k_x) = \sum_n x(n)\cos(2\pi kn/k_x) = \mathrm{Re}[X(k)]$$

and, (30)

$$y_{Qk}(k_x) = \sum_n x(n)\sin(2\pi kn/k_x) = -\mathrm{Im}[X(k)].$$

Thus, the real and imaginary components of the DFT of x(n) at frequency k are identical to the outputs of an in-phase and quadrature pair of filters, matched to frequency k and sampled at the end of the baud. Put another way, the DFT of x(n) gives us K complex outputs, or 2K real outputs, which are equivalent to the outputs of K pairs of matched

filters, each matched to one of the signals from our orthogonal set of 2K signals, sampled at the end of the signal. The linearity of the DFT insures that the response to signal plus noise of the DFT is the same as the response of the matched filter. In particular, the DFT maximizes the signal-to-noise ratio for detecting the amplitudes of the quadrature components of all the harmonics in the MFM signal. In this sense, the DFT is the optimum reciever for MFM.

III. PERFORMANCE OF MFM

Signal Received Plus Noise

In general, the signal arrives at the receiver over a link that will introduce frequency dependent attenuation and phase shifts. It may introduce a significant bulk delay (for example a fraction of a second in satellite communications and perhaps several seconds in acoustic communications). The bulk delay may vary with time due to transmitter and/or receiver dynamics introducing time and frequency dilation (generalized doppler shift). The transfer characteristics of the channel may be random in some way as in channels with fading. Inevitably noise will be added to the signal, either from the environment, the electronics or both.

In this report it will be assumed that the channel transfer function consists only of known attenuation and

that the noise is additive white guassian noise. In such a channel, synchronization may be assumed and coherent reception of the in-phase and quadrature symbols is possible as described in the previous section. The very important problems that delay and doppler shift create for synchronization and the performance of synch algorithims are being treated in a separate report. (Multipath and fading problems in MFM are the subject of a future study. They are not considered to be serious problems in the near vertical acoustic link in whose development we are currently involved. However, synchronization is a principal researh issue because of the lengthy acoustic propagation times.) Since the signals are being processed baud-wise, we shall continue to suppress the baud notation, understanding that our results apply to the "l^{th} baud" for arbitrary l. Baud identity will be reintroduced only as required for clarity.

With synchronization, the recieved signal plus noise will be sampled at f_x samples per second, and the first sample will be triggered at time u=0. This sampling will yield the discrete time signal;

$$y(n) = \sum_k (2P_k)^{\frac{1}{2}}\cos(2\pi kn/k_x + \phi_k) + w(n) \; ; \; 0 \le n \le k_x - 1 \quad (31)$$

where P_k is the received signal power of the k_{th} tone and w(n) is a sequence of white noise samples. Let us relate the statistical properties of the white noise sequence w(n)

to those of the analog continuous time white noise signal $w(u)$.

Assume that the received data $y(u)$ has been ideally bandlimited to one half the sampling frequency such that there is no power in frequencies greater than or equal to $f_x/2$. If $w(u)$ has a white PSD equal to $N_o/2$ then;

$$E[w(u)] = E[w(n)] = 0 \tag{32}$$

and,

$$\sigma^2 = Var[w(u)] = N_o f_x/2 = Var[w(n)]. \tag{33}$$

Also, sampling at twice the highest frequency assures that the noise samples of white noise are uncorrelated, i.e.,

$$E[w(n)w(m)] = 0 \quad ; n \neq m. \tag{34}$$

Signal-to-Noise Ratios (SNR's) and the DFT

Let the receiver input SNR be defined as the signal power in bandwidth W divided by the noise power in bandwidth W. Also define the average tone signal power as;

$$P_o = \sum_k P_k/K \tag{35}$$

then,

$$SNR_I = KP_o/WN_o = P_o/\Delta f N_o = P_o k_x/(2\sigma^2) . \tag{36}$$

Note that the narrowband input SNR;

$$SNR_{NB} = P_k/\Delta f N_o = P_k k_x/(2\sigma^2) \tag{37}$$

is the same as the wideband SNR of (36) if the tone power is the same as the average power, a certainty if all tones have equal received powers (Not necessarily the same as all tones having the same transmit powers if different tones suffer different attenuations. If the channel frequency response is known at least approximately, then it may be feasible to pre-emphasize the transmit tone powers such that the received signal is approximately white. Of course, this may not be practical if there are large nulls in the frequency response.)

Now consider the k_x-point DFT coefficients of the input sequence;

$$Y(k) = S(k) + W(k) \tag{38}$$

where,

$$S(k) = \tfrac{1}{2}(2P_k)^{\frac{1}{2}} k_x \exp(j\phi_k) \ ; \ k_1 \leq k \leq k_2 \tag{39}$$

and the W(k) are the DFT coefficients of the white noise sequence. It is easily shown that the W(k) have the following statistical properties;

$$E[W(k)] = 0 \tag{40}$$

$$E[\mathrm{Re}\{W(k)\}^2] = E[\mathrm{Im}\{W(k)\}^2] = k_x\sigma^2/2 \tag{41}$$

$$E[\mathrm{Re}\{W(k)\}\mathrm{Im}\{W(k)\}] = 0. \tag{42}$$

Furthermore, the Real and Imaginary parts of W(k) and W(i) are all uncorrelated with one another for $k \neq i$.

To summarize, the 2K coefficients given by the Real and Imaginary parts of the K points of the DFT corresponding to the tones used in the MFM transmit signal have means given by the Real and Imaginary parts of (39) and variances given by (40). They are uncorrelated random variables. If the input noise sequence w(n) is gaussian, they are uncorrelated gaussian random variables.

Define the output SNR of receiver as the ratio of the square of the mean to the variance for each of these 2K coefficients. These ratios are the maximum since, as we have shown previously, the DFT coefficients are identical to the output of matched filters. Thus,

$$[\mathrm{SNRI}]_k = P_k k_x (\cos\phi_k)^2/\sigma^2$$

and, (43)

$$[\mathrm{SNRQ}]_k = P_k k_x (\sin\phi_k)^2/\sigma^2$$

It is useful to note from (37) that;

$$P_k k_x/\sigma^2 = 2SNR_{NB}. \tag{44}$$

For the purpose of analysis, define the unit variance, uncorrelated random variables;

$$I_k = Re[Y(k)]/\{Var(Re[Y(k)])\}^{1/2}$$

and, (45)

$$Q_k = Im[Y(k)]/\{Var(Im[Y(k)])\}^{1/2}.$$

and note from (43) that;

$$E[I_k] = \pm\{[SNRI]_k\}^{1/2} = \{P_k k_x/\sigma^2\}^{1/2}\cos\phi_k$$

and, (46)

$$E[Q_k] = \pm\{[SNRQ]_k\}^{1/2} = \{P_k k_x/\sigma^2\}^{1/2}\sin\phi_k$$

where the $\pm$ depends on the phase ϕ_k. Furthermore, if the noise is gaussian, these random variables are unit variance, statistically independent gaussian random variables with mean values given by (46).

Bit Error Probabilities

In this section results are presented for bit error probabilities of three types of MFM: MFBPSK, MFQPSK and MF16-QAM. These modulation formats carry one, two and four bits per tone per baud. Thus they carry one, two and four

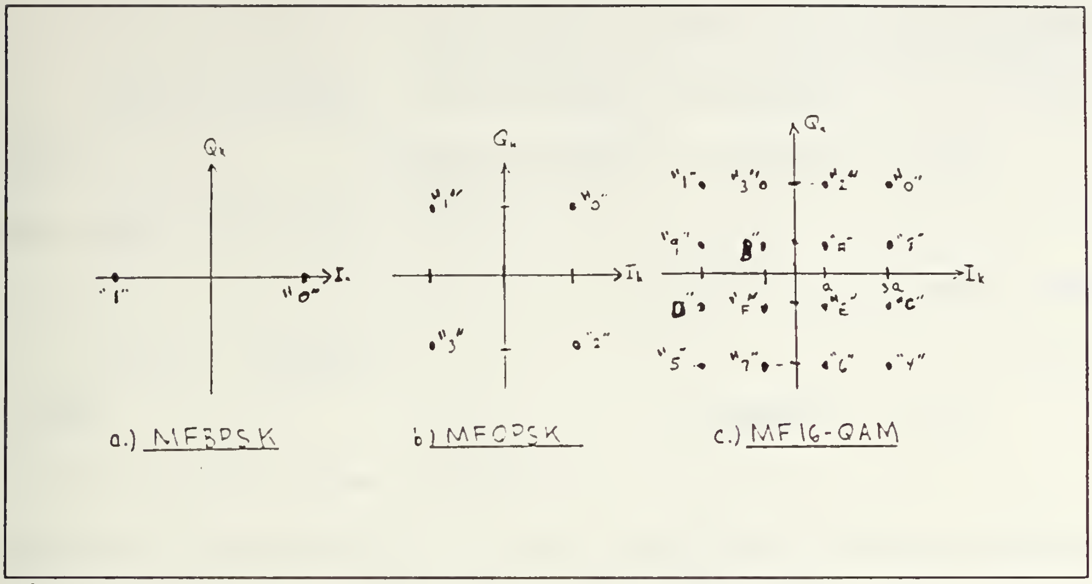

Figure 7 MFM Phase Modualation Formats (Hex)

bits per Hz of bandwidth per second or equivalently KL, 2KL and 4KL bits per packet respectively. The I_k and Q_k variables, in the absence of noise, are shown in Fig. 7 for each of the three modulation formats. The magnitudes of the tones have been choosen for equal tone powers in MFBPSK and MFQPSK and for an equal average tone power for MF16-QAM assuming each of the 16 symbols is equally probable. This makes it possible to compare modulation types under the condition of equal total (or average total) transmit power.

a. MFBPSK

In MFBPSK, the quadrature output is not used. Assuming ϕ=0, then a correct bit detection is made anytime I > 0, that is, anywhere in the right half plane of Fig. 7a. The

probability of bit error is given by;

$$P_E = 1 - \Pr[I > 0 | \ \phi=0] = Q\{E[I]\} \tag{47}$$

where from (44) and (46),

$$E[I] = \{2SNR_{NB}\}^{\frac{1}{2}} \tag{48}$$

and Q(x) is the Q-function [ref 1]. The function is bounded above by the exponential;

$$Q(x) \leq [\exp(-x^2/2)]/(2\pi)^{\frac{1}{2}} \tag{49}$$

and for x greater than two, the bound is a very good approximation to Q(x).

Eqn (47) is plotted in Fig. 8 as a function of SNR_{NB}.

b. MFQPSK

In MFQPSK, both the in phase and quadrature channels carry a bit of information. However, since they are statistically independent, we can decode them independently. For example, if $\phi = \pi/4$ corresponds to logical 00 and $\phi = -\pi/4$ corresponds to logical 10, then the least significant bit will be correctly decoded as a 0 so long as $I > 0$ when either 00 or 10 is sent. Thus, the probability of bit error is given by;

$$P_E = 1 - \Pr[I > 0 | \phi = \pm\pi/4] = Q\{E[I]\} \qquad (50)$$

where again from (44) and (46),

$$E[I] = \{SNR_{NB}\}^{\frac{1}{2}} \qquad (51)$$

Eqn (49) is also shown in Fig 8.

c. MF16-QAM

In MF16-QAM, the in phase and quadrature channels both carry two bits of information. Again, since they are statistically independent, we can decode them independently and by symmetry, the bit error probabilities calculated for one channel will be true for the other channel as well.

Let us concentrate on the in phase channel. There are four symbols; two with amplitudes $\pm(2P_L)^{\frac{1}{2}}$ and two with amplitudes $\pm(2P_H)^{\frac{1}{2}}$. We want the average power in the in phase channel to be $P_k/2$, the same as QPSK. Assuming that low and high power symbols are equally probable;

$$\tfrac{1}{2}P_L + \tfrac{1}{2}P_H = P_k/2 \qquad (52)$$

and for 16-QAM we make $(2P_H)^{\frac{1}{2}} = 3(2P_L)^{\frac{1}{2}}$ then, $P_L = P_k/10$ and $P_H = 9P_k/10$. Thus, referring to (46) we see that ;

$$E[I|P_L] = E[Re\{Y(k)|P_L]/\{Var[Re\{Y(k)\}]\}^{1/2} = (P_L k_x/\sigma^2)^{1/2} \quad (53)$$

or,

$$E[I|P_L] = \pm\{SNR_{NB}/5\}^{1/2} = \pm a \quad (54)$$

Similarly;

$$E[I|P_H] = \pm\{9SNR_{NB}/5\}^{1/2} = \pm 3a \quad (55)$$

Now referring to Fig. 7, assuming equally likely symbols, the probability of correct decoding of in phase symbols is given by;

$$P_C = \tfrac{1}{2}(\ Pr\{0 \leq I \leq 2a|P_L\} + Pr\{2a \leq I|P_H\}\ ) = 1 - 3Q(a)/2 \quad (56)$$

Since each symbol contains 2 bits, then we can define an equivalent bit error probability such that two bit sequences will have the same probability of correct decoding. That is the probability of correctly decoding two bit sequences of indepedent bits is given by;

$$P_C = (1 - P_{Eeq})^2. \quad (57)$$

Equating (55) and (56) gives an equivalent bit error probability for MF16-QAM;

$$P_{Eeq} = 1 - \{1 - 3Q(a)/2\}^{\frac{1}{2}} \tag{58}$$

where,

$$a = |E[I|P_L]| = \{SNR_{NB}/5\}^{\frac{1}{2}}. \tag{59}$$

The equivalent probability of bit error for MF16-QAM, (58) is shown in Fig. 8 as a function of SNR_{NB}.

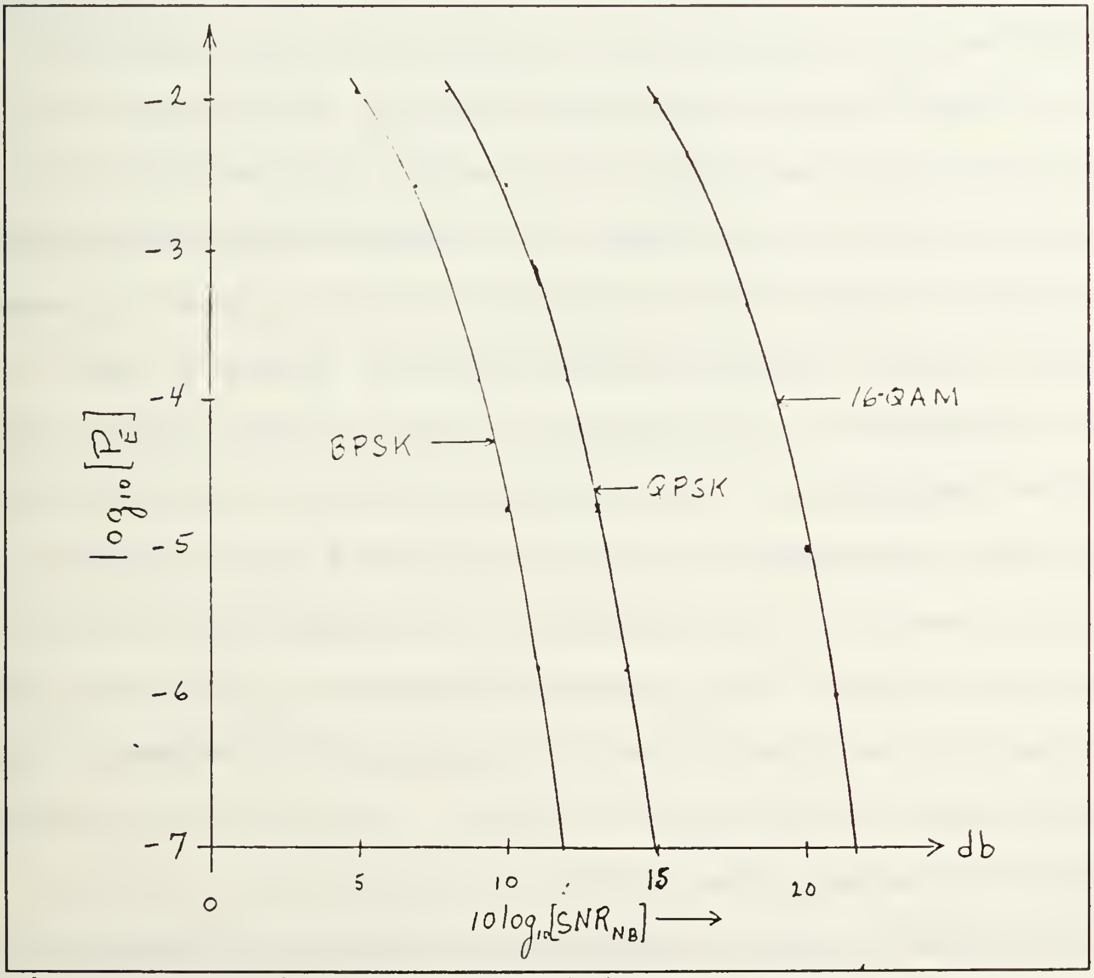

Figure 8 MFM Bit Error Probabilites vs. SNR_{NB}

Optimization

Bit error probabilities are directly reduced by increasing SNR_{NB} for MFM as illustrated clearly in the three types of modulation described above. From (37);

$$SNR_{NB} = P_k/\Delta f N_o = P_k \Delta T/N_o \quad (60)$$

with,

$$\Delta T = T/L. \quad (61)$$

Now given a fixed power for the tones and fixed additive noise level, then we want to maximize ΔT or minimize L and Δf. What restricts us from choosing L=1 and ΔT=T? Only the coherence time of the channel. If the channel is stationary over times greater than the packet length T, then the packet should consist of a single baud (L=1) of length T, tones will be spaced at 1/T Hz and there will be K=TW tones in the channel bandwidth W. If the coherence time of the channel is less than the packet length then $\Delta T \leq T_c$ and there must be $L \geq T/T_c$ bauds in the packet. As an example, for the near vertical acoustic link, it is expected that T_c will be about .025 seconds and packets are of length T=.066 seconds. Thus, packets must be made up of at least 3 bauds and tone spacing Δf must be no less than 40 Hz.

As another example, suppose we are using 10 bauds in a packet and achieving a SNR_{NB} of ten allowing transmission of data at, $10^{-4.7}$, a satifactorily low BER using MFBPSK. This

modulation provides a data rate of one bit per Hz of channel bandwidth. Now suppose the channel is sufficiently stable that it is possible to use only one baud per packet, but ten times as many tones. We can now expect an SNR_{NB} of 100 and obtain the same BER using MF16-QAM. This modulation gives a data rate of four bits per Hz of channel bandwidth, that is four times as great as the MFM using shorter bauds.

Redundancy

Redundancy may be introduced in frequency, time, or both in order to improve SNR_{NB} and reduce bit error probability. This is done at the expense of bit rate. Recall that there are 2KL orthogonal signals in a packet. If an information symbol is repeated M times on M of the 2KL signals choosen in a known pattern, the pattern may be psuedo-random for example, then the M outputs of the 2KL DFT coefficients can be added prior to decoding. This assumes that coherence is maintained amoung the M signals in transmission. Given coherence, then SNR_{NB} will be increased by $M^{\frac{1}{2}}$. Of course the bit rate will be reduced by a factor of M. This is similar to the effect introduced by 1/M rate convolutional codes [ref 7].

More will be presented on coding and on signal spreading techniques using MFM in a subsequent report. It is worthy of note at this point that redundancy can be useful to combat particular channel problems like fading or burst noise.

IV. DISCUSSION AND CONCLUSIONS

MFM is a new digital signal processing oriented communications signal ideally suited for computer to computer links and computer based information exchange networks. In this report we have presented the theory behind MFM signal generation and reception, that is modulation and demodulation, using the host computer's DSP algorithims and hardware.

Spectral and temporal properties of MFM have been derived and examples presented for bandpass and lowpass channels. Finally MFM performance, in terms of bit error rates, has been calculated for MFBPSK, MFQPSK and MF16-QAM. These signals provide, respectively, one, two and four bits per second per Hz of channel bandwidth. We have also seen that for channels with attenuation only, optimum performance, that is minimum error rate for a given data rate or maximum data rate for a given error rate, is obtained by using the minimum tone spacing and hence the maximum number of tones in the available bandwidth. The minimum tone spacing is the reciprocal of the packet length, or the coherence time of the channel, whichever is smallest.

In many channels, particuarily those where propagation over an unknown and/or time varying path is involved, there is uncertainty in the time delay and doppler dilation introduced by the propagation. This introduces the problem

of synchronization. In order to have coherent reception of MFM, synchronization must be obtained with respect to time of arrival of each baud and with respect to sampling frequency for the analog waveform, a waveform that may have been dilated by the channel. Results on synchronization error and sampling frequency error as they effect the performance of MFM will be presented in a separate NPS Technical Report. Synchronization algorithims, as well as differential MFM, a less synch sensitive form of MFM will be discussed there too.

REFERENCES

1. Couch, L. W. II, **Digital and Analog Communications Systems,** 2nd ed., Macmillan, 1987.

2. Ricci, F. J. & Schutzer, D., **U.S. Military Communications,** Computer Science Press, 1986.

3. Kalet, Irving, "The Multitone Channel", **IEEE Trans. on Communications, vol 37, no. 2**, Feb 1989, pp 119-124.

4. Moose, P. H., "Submarine Acoustic Tactical Data Link", **Proc. of MILCOM 86.**, Oct 1986, Monterey, CA.

5. Childs, Robert Daniel, "High Speed Output Interface for a Multifrequency Quatenary Phase Shift Keyed Signal Generated on an Industry Standard Computer", MSEE Thesis, Dec 1988, Naval Postgraduate School, Monterey, CA.

6. Strum, Robert D., and Kirk, Donald E., **First Principles of Discrete Signal Processing,** Addison-Wesley, 1988.

7. Lin, S., and Costello, D. J. Jr., **Error Control Coding,** Prentice Hall, 1983.

APPENDIX 1
DESIGN PARAMETERS FOR 1/15TH SECOND SIGNAL PACKET IN A 16-20KHZ BANDPASS CHANNEL

ΔT sec	1/240	1/120	1/60	1/30	1/15
L	16	8	4	2	1
Δf	240	120	60	30	15
k_1	68	135	269	537	1073
f_1	16320	16200	16140	16110	16095
k_2	83	166	332	664	1328
f_2	19920	19920	19920	19920	19920
k_x	256	512	1024	2048	4096
f_x	61440	61440	61440	61440	61440

DISTIBUTION LIST

		No. Copies
1.	Defense Technical Information Center Cameron Station Alexandria, VA 22304-6145	2
2.	Library, Code 0142 Naval Postgraduate School Monterey, CA 93943-5002	2
3.	Department Chairman, Code 62 Naval Postgraduate School Monterey, CA 93943-5004	1
4.	Director of Research Administration, Code 012 Naval Postgraduate School Monterey, CA 93943-5000	1
5.	Paul H. Moose, Assoc. Prof, Code 62me Naval Postgraduate School Monterey, CA 93943-5004	12
6.	Darrell Marsh, Code 624 Naval Ocean Systems Center San Diego, CA 92152	2